Linear Algebra

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If you are unfamiliar with linear or matrix algebra, you will find that it is very different from basic algebra or calculus. For the duration of this session, we will be focusing on definitions of such concepts as linear equations, matrices, determinants, vector spaces, inner products, linear transformations, eigenvalues and eigenvectors, and their applications to interesting mathematical problems.

Texts:

Linear Algebra and Its Applications (3rd Edition) Addison Wesley ©2003, by David C. Lay (DCL)

Module 1

Properties of Matrices
System of Linear Equation
DCL (Recommended):
1.1.16, 22, 30
1.6.8, 14, 15
1.7.1, 3, 5, 11
1.8.1, 3, 15, 21

Module 2

Linear Independence
Inverse of Matrix Linear Systems and Inverses
Determinants
DCL(Recommended):
2.1.3, 5, 7, 10, 15, 16, 19, 25
3.1.1, 9, 11, 19, 21
3.2.3, 9, 10

Module 3

Eigenvalues/Eigenvectors
Cramer's Rule
Orthogonality(?)
DCL(Recommended):
5.1.3, 4, 5, 7, 8
5.2.7, 9
5.3.3, 11, 13
6.1.3, 5, 7
6.2.5, 7
6.3.3, 9, 11

1 Linear Algebra: Module 1

1.1 Module Topics:[1]

System of Linear Equations
Method of Substitution
Gaussian Elimination
Gauss-Jordan Elimination
Matrix Methods

1.1.1 Linear Equation

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

$4x_1 - 5x_2 + 2 = x_1 \rightarrow rearranged \rightarrow$

$x_2 = 2(\sqrt{6} - x_1) + x_3 \rightarrow rearranged \rightarrow$

1.1.2 Systems of Linear Equations

Collection of one or more linear equations involving Collection of one or more linear equations involving same variables – say $x_1, ..., x_n$. An example is

$$\begin{gathered} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{gathered}$$

More generally, we might have a system of m equations in n unknowns

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{gathered}$$

[1] Examples for this workshop were drawn from Lay (2003) *Linear Algebra and Its Applications*, Leon(2002) *Linear Algebra with Applications* (6th Edition), and Harvard University's Political Science Math (P)refresher.

The set of all possible solutions is called the *solution set* of the linear system.

Two linear systems are called *equivalent* if they have the same solution sets.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines.

$$x_1 + x_2 = 10 \qquad\qquad x_1 - 2x_2 = -3 \qquad\qquad x_1 + x_2 = 3$$
$$-x_1 + x_2 = 0 \qquad\qquad 2x_1 - 4x_2 = 8 \qquad\qquad -2x_1 - 2x_2 = -6$$

If there are three equations in three variables, each equation would define a plane in a 3 dimensional space. Same reasoning applies. If there are more than 3 variables, the intersection of hyperplanes would determine the solution set.

1.1.3 Strategies for Solving a System

In order to solve the system of linear equations, we can utilize all or one of the following three strategies:

1. Substitution
2. Elimination (of variables)
3. Matrix Methods

Method of Substitution Steps:

1. Solve one equation for one variable, say x_1, in terms of the other variables in the equation
2. Substitute the expression for x_1 into the other $m-1$equations, resulting in a new system of $m-1$ equations in $n-1$unknowns.
3. Repeat steps 1 & 2 until one equation in one unknown, say x_n. We now have a value for x_n.
4. Backward substitution: substitute x_n into previous equation(s). Repeat using the successive expressions of each variable in terms of the other variables to find the values of all x_i's.

Using substitution, solve:

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$

Using substitution, solve:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Method of Elimination a) Elementary Row Operations

Elementary row operations are used to transform the equations of a linear system, while maintaining an equivalent linear system – equivalent in the sense that the same values of xj solve both the original and transformed systems. These operations are

1. (Replacement) Add one row to a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Take Example 2, for instance, we can:
Add the 1st row to the 2nd row of equation to get:
$x_1 - 2x_2 = -1$
$x_2 = 2$
Then use backward substitution to get
$x_1 = 3$
$x_2 = 2$

We can apply the similar procedure to solve for Example 3

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

(i) To do so, add 4 times 1st row to the 3rd row of equation. To get a new 3rd row of equation.

(ii) Now, multiply equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmetic for the next step.)

(iii) Use the x_2 in equation 2 to eliminate the $-3x_2$ in equation 3 in order to obtain:

(iv) The new system has a triangular form:

We can use backward substitution to find the solution set.

Check to see if (29, 16, 3) is a solution of the original system.

b) Gaussian Elimination

Method by which we start with some linear system of m equations in n unknowns and use the elementary equation operations to eliminate variables until we arrive at an equivalent system of the form:

$$\begin{aligned}
\mathbf{a}'_{11}\mathbf{x}_1 + a'_{12}x_2 + a'_{13}x_3 + ... + a'_{1n}x_n &= b'_1 \\
\mathbf{a}'_{22}\mathbf{x}_2 + a'_{23}x_3 + ... + a'_{2n}x_n &= b'_2 \\
\mathbf{a}'_{33}\mathbf{x}_3 + ... + a'_{3n}x_n &= b'_3 \\
\vdots \qquad\qquad & \vdots \\
\vdots \qquad\qquad & \vdots \\
\vdots \qquad\qquad & \vdots \\
\mathbf{a}'_{mn}\mathbf{x}_n &= b'_m
\end{aligned}$$

The bold faced coefficients are referred to as *pivots.*

(Essentially similar to the method shown above)

c) Gauss-Jordan Elimination

Takes the Gaussian elimination method one step further. Once the linear system is in the reduced form shown in the preceding section, elementary row operations and Gaussian eliminations are used to

(i) Change the coefficient of the pivot term in each equation to 1

and

(ii) Eliminate all terms above each pivot in its column,

resulting in a reduced, equivalent system. For a system of m equations in n unknowns, a typical reduced system would be:

$$\begin{array}{lllllll} x_1 & & & & & & = b_1^* \\ & x_2 & & & & & = b_2^* \\ & & x_3 & & & & = b_3^* \\ & & & . & & & \vdots \\ & & & & . & & \vdots \\ & & & & & . & \vdots \\ & & & & & & x_n = b_m^* \end{array}$$

d) Matrix Method

Matrices provide an easy and efficient way to represent linear systems such as:

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array}$$

The $m \times n$ **coefficient matrix A** is an array of mn real numbers arranged in m rows by n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & . & & a_{2n} \\ \vdots & & . & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

The RHS of the linear system is represented by the vector $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Augmented matrix: when we append b to the coefficient matrix A, we get the augmented matrix $\hat{A} = [A|b]$

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & . & & a_{2n} & b_2 \\ : & & . & : & : \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{pmatrix}$$

The rules of elementary row operations apply here.

Gaussian elimination of the augmented matrix results in a **row echelon form**.

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ 0 & a_{22} & a_{23} & a_{2n} & b_2 \\ : & 0 & & : & : \\ 0 & 0 & 0 & a_{mn} & b_m \end{pmatrix}$$

Note:

(i) All nonzero rows are above any rows of all zeros.

(ii) Each leading entry of a row is in a column to the right of the leading entry of the row above it.

(iii) All entries in a column below a leading entry are zeros.

Reduced row echelon form

Satisfies the following additional conditions

(iv) The leading entry in each nonzero row is 1.

(v) Each leading 1 is the only nonzero entry in its column.

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & b_1 \\ 0 & a_{22} & 0 & 0 & b_2 \\ : & 0 & ... & & : \\ 0 & 0 & 0 & a_{mn} & b_m \end{pmatrix}$$

In essence, this is the matrix equivalent of a linear system after Gauss-Jordan elimination.

Exercise 1 *Solve the system using matrix method:*

$x_1 + x_2 = 1$

$x_1 - x_2 = 3$

$-x_1 + 2x_2 = -2$

Exercise 2 *Solve the system using matrix method*

$-x_1 + x_2 - x_3 + 3x_4 = 0$
$3x_1 + x_2 - x_3 - x_4 = 0$
$2x_1 - x_2 - x_3 - x_4 = 0$

Exercise 3 *In the downtown section of a certain city, two sets of one way streets intersect as shown below. The average hourly volume traffic entering and leaving this section during rush hour is given in the diagram. Determine the amount of traffic between each of the four intersections.*

Rank, Uniqueness and Consistency As the exercises suggest, there are two fundamental questions that we may want to address when we are solving for a system of linear equations

1) Is the system consistent? (Does a solution exist?)
2) If a solution exists, is it unique? (Is there one and only one solution?)

We can determine whether one infinite, or no solutions exist if we know (1) the number of equations m, (2) the number of unknowns n, and (3) the rank of the matrix representing the linear system.

Rank: number of nonzero rows in its row echelon form.

Example 4 $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ $\qquad Rank = 3$

Example 5 $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad Rank = 2$

Example 6 $\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 4 & 5 & b_2 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & b_4 \end{pmatrix}, b_j \neq 0$ $\qquad Rank = 4$

Let A be the coefficient matrix and $\hat{A} = [A|b]$ be the augmented matrix. Then,

1. rank A $\leq$rank Â

   Augmenting A with b can never result in more zero rows than originally in A itself. Suppose row i in A is all zeros and that b_i is non-zero. Augmenting A with b will yield a non-zero row i in $\hat{A}$.

2. rank A $\leq$ rows A

   By definition of a rank.3

3. rank A $\leq$ columns A

   Suppose there are more rows than columns (otherwise the previous rule applies). Each column can contain at most one pivot. By pivoting, all other entries in a column below the pivot are zeroed. Hence, there will only be as many non-zero rows as pivots, which will equal the number of columns.

Existence of solutions:

1. Exactly one solution

   rank A = rank Â = rows A = columns A

   Necessary condition for a system to have a unique solution: that there be exactly as many equations as unknowns.

2. Infinite solutions

   rank A = rank; and columns A > rank A

   If a system has a solution and has more unknowns than equations, then it has infinitely many solutions.

3. No solution

   rank A < rank Â

   Then there is a zero-row i in A's reduced echelon that corresponds to a non-zero row i in $\hat{A}$'s reduced echelon. Row i of the translates to the equation:

$$0x_{i1} + 0x_{i2} + ... + 0x_{in} = b_i$$

Where $b_i \neq 0$. Hence, the system has no solution.

Exercise 7 $x_1 + 2x_2 + x_3 = 1$
$2x_1 + 4x_2 + 2x_3 = 3$

Exercise 8 $x_1 + x_2 + x_3 + x_4 + x_5 = 2$
$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$
$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$

2 Linear Algebra: Module 2

2.1 Module Topics:[2]

Linear Independence
Matrix Algebra
Inverse
Determinants
Cramer's Rule
Eigenvalues & Eigenvectors

2.1.1 Linear Independence

Let $v_1, v_2, ..., v_n$ be a set of n vectors each of which is of order m. Then the set of vectors is linearly dependent if there exist scalars $a_1, a_2, ..., a_n$ at least one of which is not 0 such that

$$a_1v_1 + a_2v_2 + a_3v_3 + ... + a_nv_n = 0$$

Or

[2] Examples for this workshop were drawn from Lay (2003) *Linear Algebra and Its Applications*, Leon(2002) *Linear Algebra with Applications* (6th Edition), and Harvard University's Political Science Math (P)refresher.

$$Va = 0$$

Example 9 *Is there linear dependence?*

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Hint: We can try to augment these vectors into a matrix and solve for a reduced row echelon form and see if we can find some relationship between the rows. If we can determine some functional linear relationship between these vectors, we can say that there is linear dependence.

2.1.2 Matrix Algebra

Matrix: A matrix is an array of mn real numbers arranged in m rows by n column.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ : & & & : \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

Matrix addition (or subtraction): Let A and B be two $m \times n$ matrices. Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & ... & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & ... & a_{2n} + b_{2n} \\ : & & & \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & .... & a_{mn} + b_{mn} \end{pmatrix}$$

Note: A and B must be the same size!

Example 10 $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} =$

Example 11 $\begin{pmatrix} 3 & 6 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 5 & 6 \end{pmatrix} =$

Example 12 $\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} =$

Scalar Multiplication: Given the scalar s, the scalar multiplication of sA is

$$sA = s\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} = \begin{pmatrix} sa_{11} & sa_{12} & ... & sa_{1n} \\ sa_{21} & sa_{22} & & sa_{2n} \\ \vdots & & & \vdots \\ sa_{m1} & sa_{m2} & ... & sa_{mn} \end{pmatrix}$$

Matrix multiplication: If A is an $m \times n$ matrix and B is a $k \times n$ matrix, then their product $C = AB$ is an $m \times n$ matrix where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{ik}b_{kj}$$

So

$$\begin{pmatrix} 3 & -2 \\ 2 & 4 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{pmatrix} = \begin{pmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ 10 & 4 & 21 \end{pmatrix}$$

If $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{pmatrix}$ then, $BA =$?

Note: the number of columns of the first matrix must equal the number of rows of the second matrix. The size of the matrices (including the resulting product) must be:

$$(m \times k)(k \times n) = (m \times n)$$

Using matrix multiplication, we can now write the linear equation.

Let A be a $m \times n$ coefficient matrix $\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$

x be a $m \times 1$ vector array of unknowns $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

b be a $m \times 1$ vector array of constants $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Then,

$$
\begin{gathered}
Ax = b \\
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_m = b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_m = b_2 \\
: \\
: \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_m = b_m
\end{gathered}
$$

2.1.3 Law of Matrix Algebra

1. Associative

   $(A + B) + C = A + (B + C)$

   $(AB)C = A(BC)$

2. Commutative

   $A + B = B + A$

   Question: Is multiplication of matrices commutative? $AB \neq BA$ or $AB = BA$?Can you think of an example that can show whether one or the other?

3. Distributive

   $A(B + C) = AB + AC$

   $(A + B)C = AC + BC$

Exercise 13 *If* $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Solve
a) $A(BC)$
b) $(AB)C$
c) $A(B + C)$
d) $AB + AC$

If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ then, $A^2 =? \; A^3 =? \; A^n =?$

1. Transpose

   Transpose of the $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A.

The following rules apply for transposed matrices
a) $(A + B)^T = A^T + B^T$
b) $(A^T)^T = A$
c).$(sA)^T = sA^T$

Exercise 14 $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{pmatrix}$ *Solve for $B^T A^T$ and $(AB)^T$*

d) $(AB)^T = B^T A^T$

2.1.4 The Inverse of a Matrix

Identity Matrix: The $n \times n$ identity matrix I_n is the matrix whose diagonal elements are 1 and all off-diagonal elements are 0. So

$$I_n = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$ where all diagonal $a_{11} = a_{22} = a_{33} = ... = a_{mn} = 1$ and all other off-diagonal elements are 0.

Inverse Matrix: $n \times n$ matrix A is nonsingular or invertible if there exists an $n \times n$ matrix A such that

$$AA^{-1} = A^{-1}A = I_n$$

A^{-1} is the inverse of A. If there is no such A^{-1}, then A is singular or non-invertible.

Invertible:

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (2.1)$$

If $ad - bc \neq 0$, then A is invertible (or nonsingular)

$$\text{and } A^{-1} = \tfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Properties of Inverse

1. If the inverse exists, it is unique
2. A nonsingular $\rightarrow A^{-1}$ nonsingular $\quad (A^{-1})^{-1} = A$
3. A and B nonsingular $\rightarrow$ AB nonsingular $\quad (AB)^{-1} = B^{-1}A^{-1}$

4. A nonsingular $\rightarrow (A^T)^{-1} = (A^{-1})^T$

Exercise 15 *Find* A^{-1} *if*

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

Determinants

If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define,

$$\det(A) = a$$

then A will be nonsingular if and only if $\det(A) \neq 0$. Hence, the determinant will equal zero when the inverse does not exist. Since the inverse of a, $\frac{1}{a}$, does not exist when $a = 0$, we let the determinant of a be a.

2×2 Matrix

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

In order to find its determinant,

1. Multiply the second row of A by a_{11}

2. Subtract a_{21} times the first row from the new second row

There are two alternatives to this method:

a) Take the first upper left hand corner element and eliminate its corresponding row and column elements; then multiply with the remaining element(s). Do the same for all other first row elements alternating subtraction and addition.

b) Simply multiply the diagonal elemtns together and subtract the product of the upper left hand element in the matrix to that of the other.

The determinant is of the form: $a_{11}a_{22} - a_{12}a_{21} \neq 0$

We say that A is nonsingular only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. We then define the determinant of a 2×2 matrix A as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}|a_{22}| - a_{12}|a_{21}|$$

Extending this to a 3×3 matrix we get

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is given by
$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + ... + (-1)^{1+n} a_{1n} \det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$

For a triangular or diagonal matrices, the determinant is just the product of the diagonal terms.

Example: $$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

Then $|R| = r_{11} \begin{vmatrix} r_{22} & r_{23} \\ 0 & r_{33} \end{vmatrix} = r_{11}r_{22}r_{33}$

Exercise 16 $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$

Exercise 17 $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Shortcut for a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11})$$

Properties of determinants

1. $|A| = |A^T|$
2. If B results from A by interchanging two rows, then$|B| = |-A|$
3. If two rows of A are equal, then $|A| = 0$
4. If a row of A consists of all zeros, then $|A| = 0$
5. If B is obtained by multiplying a row of A by a scalar s, then $|B| = |sA|$
6. If B is obtained from A by adding to the ith row of A the jth row $(i \neq j)$ multiplied by a scalar s, then $|B| = |A|$
7. If no row interchanges and no scalar multplicatinos of a single row are used to compute the row echelon form R from the $n \times n$ coefficient matrix A, then $|A| = |R|$
8. A square matrix is nonsingular if and only if its determinent is not zero
9. $|AB| = |A| \times |B|$
10. If A is nonsingular, then $|A| \neq 0$ and $|A^{-1}| = \frac{1}{|A|}$

Knowing the determinant is useful. For instance, we can calculate the matrix inverse if we know its determinant:

$A^{-1} = \frac{1}{|A|} adj(A)$ where $adj(A)$ [adjoint of A] is a $n \times n$ matrix whose (i,j)th entry is Cji.

So, if we see equation 2.1,

$$adjA = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

For a 3×3 matrix, the adjoint is the transpose of the *cofactors*

If $B = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ $\qquad adj(B) =$

Using above definition of an inverse, calculate B^{-1}

$$\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}^{-1} =$$

2.1.5 Cramer's Rule

Let A be a $n \times n$ nonsingular matrix and let $b \in R^n$. Let A_i be the matrix obtained by replacing the ith column of A by b. If x is the unique aolution to $Ax = b$, then

$$x_i = \frac{\det(A_i)}{\det(A)} \text{ for } i = 1, 2, ..., n$$

Use Cramer's Rule to solve

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 2x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 9$$

Note: Cramer's rule gives us a convenient method for writing the solution to a $n \times n$ system of linear equations in terms of determinants. However, we must evaluate n+1 determinents of order n if we wish to use Cramer's Rule. Evaluating even two of these determinants generally involves more computation than solving the system using Gaussian elimination technique.

2.1.6 Eigenvalues & Eigenvectors

Consider the following:

Given a $k \times k$ matrix A, can we find a scalar λ and a nonzero vector x such that $Ax = \lambda x$?

The answer to this question is a "yes". In fact, there exists k (possibly nondistinct) values for λ and infintely many vectors x satisfying this condition.

This is known as the eigenvalue/eigenvector problem.

How do we calculate them?

2.1.7 Some useful application of linear algebra: Linear Regression

Although linear algebra may seem impractical for first time users, it is a very useful tool for understanding the theoretical basis of linear regression. To illustrate, if we have a collection of data where the dependent variable is y and explanatory variables are series of X's, we can represent them using matrix notations as:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix}$$

If we wish to represent the relationship between these two variables y and X using linear regression, they are:

$y = X\beta + \varepsilon$

where β represents the coreponding coefficient vector based on the data y and X, and ε is the random error term.

The most popular method of estimating β is by using the *least squares* method where we seek to minimize the sum of the squared elements of ε. That is, we seek to minimize

$$\sum_{i=1}^{n} \varepsilon_i^2 = \varepsilon^T \varepsilon = (y - X\beta)^T (y - X\beta)$$

One way to do this is to find the derivative of the above equation with respect to β and then set it equal to zero and solve for β.

This yields:

$$X^T X\beta = X^T y$$

If we minimize this term, we can obtain the coefficient estimate $\hat{\beta}$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

So we can actually try this with real data to see if it works.

Let y be the price in thousands of dollars
Let x_1 be the lot size (in hundreds of square yards).
Let x_2 be the number of bedrooms

$$y = \begin{pmatrix} 122 \\ 115 \\ 145 \end{pmatrix} \qquad X = \begin{pmatrix} 10 & 4 \\ 12 & 3 \\ 14 & 5 \end{pmatrix}$$

Find the linear relationship between y and X.